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UPC condition in polynomially bounded o-minimal structures

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Abstract

We prove that UPC condition holds in o-minimal structures generated by some quasi-analytic classes of $\mathscr{C}^\infty$ functions. We also give a sufficient and necessary condition for a bounded set $A \subset \mathbb{R}^2$ definable in some polynomially bounded o-minimal structure to be UPC.

Keywords: o-Minimal structure; Definable set; Łojasiewicz's inequality; UPC set; Markov's inequality

1. Introduction

In [9], Pawłucki and Pleśniak introduced the notion of an uniformly polynomially cuspidal (UPC) set. Recall that $E \subset \mathbb{R}^n$ is called UPC if there exist $m, M > 0$ and a positive integer d such that for each $x \in \overline{E}$ we can choose a polynomial map $h_x : \mathbb{R} \longrightarrow \mathbb{R}^n$ of degree at most d satisfying the following conditions:

(1) $h_x(0) = x$,
(2) $\operatorname{dist}(h_x(t), \mathbb{R}^n \setminus E) \geqslant Mt^m$ for all $t \in [0, 1]$.

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(Note that a UPC set E is fat—that is $\overline{E} = \overline{\text{Int}\, E}$.) The importance of UPC property lies in the fact that it is a geometric sufficient condition for Markov's inequality—that was proved by Pawłucki and Pleśniak. Other applications can be found in [9,10]. Pawłucki and Pleśniak proved as well that each bounded, fat and subanalytic set is UPC. (Detailed study of subanalytic sets can be found in [1].) Their approach involved two important tools: Hironaka's rectilinearization theorem and Łojasiewicz's inequality. The first purpose of this paper is to generalize the main result of Pawłucki and Pleśniak [9] to some particular o-minimal structures, namely these o-minimal structures that are considered in [12]. (See [4,5] for the definition and properties of o-minimal structures.) Given a class C of $\mathcal{C}^\infty$ functions satisfying some properties (the most important is quasi-analyticity) Rolin et al. [12] constructed a polynomially bounded o-minimal structure $\mathbb{R}_C$. In Section 2 we prove UPC condition for all bounded, fat and definable sets in $\mathbb{R}_C$.

Of course, we are interested in UPC property in general o-minimal structures. We will say that UPC condition holds in some o-minimal structure if A is UPC whenever A is bounded, definable and fat. The problem is to characterize o-minimal structures for which UPC condition holds. The related question is to characterize UPC definable sets. Clearly, UPC condition cannot hold in an o-minimal structure which is not polynomially bounded, because by the growth dichotomy the set $E := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 \leqslant 1,\ 0 \leqslant x_2 \leqslant \exp(-x_1^{-1})\}$ is definable in such structure (cf. [7]). Thus we restrict ourselves to polynomially bounded o-minimal structures. But UPC condition may not hold, even though the structure is polynomially bounded.

Example 1. Let $A = \{(x_1, x_2) \in \mathbb{R}^2 \mid f(x_1) \leqslant x_2 \leqslant g(x_1),\ x_1 \in [0,\ 1]\}$, where $f(u) = \sum_{i=1}^{\infty} \frac{1}{2^i} u^{1-\frac{1}{i}}$ for $u \in [0,\ 1]$ and $g(u) = f(u) + u$. One easily verifies that A is not UPC, but it is definable in some polynomially bounded o-minimal structure (cf. [6]).

Example 2. Let $B = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^{\sqrt{2}} \leqslant x_2 \leqslant x_1^{\sqrt{2}} + x_1^2,\ x_1 \in [0,\ 1]\}$. Then B is definable in a polynomially bounded o-minimal structure (cf. [6]), but it is not UPC.

Dealing with general definable sets we moreover restrict ourselves to dimension two. It seems that the case of higher dimensions is much more difficult. The second main result of this paper is a characterization of bounded and plane UPC sets definable in polynomially bounded o-minimal structures (Section 4). Taking into account that UPC property implies Markov's inequality, one should say that this is connected with the paper of Goetgheluck [3] who has first proved Markov's inequality on some particular subsets of $\mathbb{R}^2$ with cusps.

We conclude this section with stating the two main results:

Theorem A. *UPC condition holds in* $\mathbb{R}_C$.

Theorem B. *Let* $A \subset \mathbb{R}^2$ *be bounded and definable in some polynomially bounded o-minimal structure. Then A is UPC if and only if A is fat and the following condition is* *satisfied*: *for each* $a \in \overline{A}$, $r > 0$ *and any connected component S of the set* $\mathrm{Int}\, A \cap B(a, r)$ *such that* $a \in \overline{S}$ *there is a polynomial arc* $\gamma : (0, 1) \longrightarrow S$ *such that* $\lim_{t\to 0} \gamma(t) = a$, *where* $B(a, r) = \{x \in \mathbb{R}^2 : \|x - a\| < r\}$.

2. UPC Property in $\mathbb{R}_C$

For $r = (r_1, \ldots, r_n) \in (0, \infty)^n$ put $I_r = (-r_1, r_1) \times \cdots \times (-r_n, r_n)$. Suppose that for every compact box $B = [a_1, b_1] \times \cdots \times [a_n, b_n]$, where $a_i < b_i$ for $i = 1, \ldots, n$ and $n \in \mathbb{N}$, we have an $\mathbb{R}$-algebra C_B of functions $f : B \longrightarrow \mathbb{R}$ satisfying the properties listed on p. 762 in [12]. Recall only the most important one—quasi-analyticity:

$T_0 : C_n \longrightarrow \mathbb{R}[[X]]$ is a monomorphism of $\mathbb{R}$-algebras, where C_n is the collection of germs at the origin of functions from $\bigcup\{C_{n,r} : r \in (0, \infty)^n\}$ and $T_0(f)$ is the Taylor series of f at the origin ($C_{n,r} := C_{\overline{I_r}}$).

Let $\mathcal{F} = \bigcup\{C_{n,1} : n \in \mathbb{N}\}$ and put $\mathbb{R}_C = \mathbb{R}(\mathcal{F})$. (If $\overline{\varepsilon} = (\varepsilon, \ldots, \varepsilon) \in (0, \infty)^n$, then we write $C_{n,\varepsilon}$ instead of $C_{n,\overline{\varepsilon}}$.)

Theorem 2.1 (*Rolin et al. [12]*). *The structure* $\mathbb{R}_C$ *is model complete, o-minimal and polynomially bounded.*

Proof. Cf. [12, Theorems 5.2, 5.4]. □

We say that a map $f \in C_{n,r}$ is NC if $f(x) = x_1^{\alpha_1} \ldots x_n^{\alpha_n} g(x),\ g \in C_{n,s},\ s > r$ ($s_i > r_i$ for $i = 1, \ldots, n$), $g(x) \neq 0$ for each $x \in I_s$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$.

The following theorem is due to Bierstone and Milman:

Theorem 2.2. *Suppose that* $f \in C_{n,\varepsilon}$ *and* $f \neq 0$. *Then there is a family* $\{\Pi_j\}$ *of mappings such that, for each j,* $\Pi_j \in (C_{n,r_j})^n$, $\Pi_j(\overline{I_{r_j}}) \subset I_\varepsilon$, $f \circ \Pi_j$ *is NC and if* $0 < s_j \leqslant r_j$ *(as polyradii), then the union of some finite subfamily of the family* $\{\Pi_j(\overline{I_{s_j}})\}$ *is a neighbourhood of the origin.*

Proof. Cf. [2, Theorem 2.4]. The theorem is also a simple consequence of Theorem 2.5 in [12]. □

A set $A \subset \mathbb{R}^n$ is called a *basic C-set* if there are $f, g_1, \ldots, g_k \in C_{n,r}, r \in (0, \infty)^n$ such that $A = \{x \in I_r \mid f(x) = 0, g_1(x), \ldots, g_k(x) > 0\}$. A finite union of basic C-sets is called a *C-set*. The set *A* is called *C-semianalytic* if for each $a \in \mathbb{R}^n$ there is an $r \in (0, \infty)^n$ such that $(A - a) \cap I_r$ is a C-set.

Theorem 2.3. *Let* $E \subset \mathbb{R}^n$ *be C-semianalytic and let* $K \subset \mathbb{R}^n$ *be compact. Then there is a finite family of mappings* $\Pi_j \in (C_{n,1})^n$ *such that* $\bigcup_j \Pi_j([-1, 1]^n)$ *is a neighbourhood of the set K and, for each j,* $\Pi_j^{-1}(E)$ *is a union of sets of the form*

$$\{x \in [-1, 1]^n : \operatorname{sgn} x_i = \sigma_i\},\ \sigma \in \{-1, 0, 1\}^n.$$

Proof. Note first that it is enough to prove the theorem in the case when K is a point. Moreover, we can assume that $K = \{0\}$. For some $\varepsilon \in (0, \infty)^n$ we have

$$E \cap I_\varepsilon = \bigcup_k \{x \in I_\varepsilon : f_k(x) = 0, \ g_{ik}(x) > 0\},$$

where $f_k, g_{ik} \in C_{n,\varepsilon}$ and without loss of generality they are not identically zero. Put $f = \prod_k f_k \cdot \prod_{i,k} g_{ik}$. Now we apply Theorem 2.2. We may assume that each I_{r_j} is equal to $(-1, 1)^n$ and that each $f_k \circ \Pi_j, \ g_{ik} \circ \Pi_j$ is NC. The rest of the proof is now straightforward. □

Theorem 2.4 (*cf. Pawłucki and Pleśniak [9, Corollary 6.2]*). *Let $F \subset \mathbb{R}^n$ be bounded and C-semianalytic and let $E = \Pi(F) \subset \mathbb{R}^k$, where $\Pi : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is the projection onto the subspace of first k coordinates. Then there exists a finite family of mappings $\Psi_j \in (C_{n_j,1})^k, \ j = 1, \ldots, s$, such that $\Psi_j((-1, 1)^{n_j}) \subset E$ for all j and $\bigcup_{j=1}^{s} \Psi_j([-1, 1]^{n_j}) = \overline{E}$.*

Proof. We can assume that E has no isolated points. Applying Theorem 2.3 to F and $K = \overline{F}$ we obtain a family $\{\Pi_j\}, \ j = 1, \ldots, s$. For each j, $\Pi_j^{-1}(F)$ is the union of sets $T_{j\alpha} = \{x \in [-1, 1]^n : \operatorname{sgn} x_i = \alpha_i\}$ with some $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{-1, 0, 1\}^n$. For all $T_{j\alpha} \neq \{0\}$, take $H_{j\alpha} := \operatorname{Int} T_{j\alpha}$, where the interior is taken in the linear span of the set $T_{j\alpha}$. Let $\Psi_{j\alpha} = \Pi \circ \Pi_j|_{\overline{H_{j\alpha}}}$. Clearly, $\bigcup \Psi_{j\alpha}(H_{j\alpha}) \subset E$ and $\bigcup \Psi_{j\alpha}(\overline{H_{j\alpha}}) = \overline{E}$. We may assume that $H_{j\alpha} = (-1, 1)^{n_{j\alpha}}$. □

Theorem 2.5. *Suppose that an open set $\Omega \subset \mathbb{R}^k$ is the projection onto $\mathbb{R}^k$ of some C-semianalytic and bounded subset of $\mathbb{R}^n$. Then Ω is UPC.*

Proof. We follow the proof of Theorem 6.4 in [9]. Fix a positive integer p and let

$$g_p : [-1, 1]^p \times [0, 1] \ni (x, t) \mapsto (x_1(1-t), \ldots, x_p(1-t)) \in [-1, 1]^p.$$

Note that $g_p([-1, 1]^p \times (0, 1]) \subset (-1, 1)^p$ and $g_p([-1, 1]^p \times \{0\}) = [-1, 1]^p$. By the previous theorem, there exist $\Psi_j \in (C_{n_j,1})^k, \ j = 1, \ldots, s$, such that

$$f_j([-1, 1]^{n_j} \times (0, 1]) \subset \Omega, \quad \bigcup_{j=1}^{s} f_j([-1, 1]^{n_j} \times \{0\}) = \overline{\Omega},$$

where $f_j = \Psi_j \circ g_{n_j}$. By the Łojasiewicz inequality, there are $C, m > 0$ such that $\operatorname{dist}(f_j(x, t), \mathbb{R}^k \setminus \Omega) \geqslant Ct^m$ for $x \in [-1, 1]^{n_j}$ and $t \in [0, 1]$. Take a positive integer $d \geqslant m$ and fix j. We have

$$f_j(x,t) = \sum_{\ell=0}^{d} \frac{t^\ell}{\ell!} \frac{\partial^\ell f_j}{\partial t^\ell}(x,0) + t^{d+1} Q_j(x,t),$$

where Q_j is $\mathcal{C}^\infty$ in a neighbourhood of the set $[-1,\ 1]^{n_j} \times [0,\ 1]$. Choose $\delta \in (0,\ 1]$ in a way such that $\|t Q_j(x,t)\| \leqslant \frac{C}{2}$ for $(x,t) \in [-1,\ 1]^{n_j} \times [0,\ \delta]$. Then

$$\mathrm{dist}(f_j(x,\delta t) - \delta^{d+1} t^{d+1} Q_j(x,\delta t),\ \mathbb{R}^k \setminus \Omega) \geqslant \frac{C\delta^m}{2} t^m$$

for $x \in [-1,\ 1]^{n_j},\ \ t \in [0,\ 1]$. The end of the proof is now obvious. □

Proof of Theorem A. It follows from the way Theorem 2.1 is proved in [12] that each bounded and definable set in $\mathbb{R}_C$ is the projection of some bounded and C-semianalytic set. Thus it is enough to use Theorem 2.5. □

Remark. Theorem A gives along with Theorem 3.1 in [9] the positive answer to Question 3.8 in [11] posed by Pleśniak and concerning Markov's inequality.

3. A necessary condition for UPC property

We will say that an o-minimal structure $\mathcal{S}$ admits polynomial curve selection if for each open and definable set Ω in $\mathcal{S}$ and for each $a \in \overline{\Omega}$ there is a polynomial arc $\gamma : (0,\ 1) \longrightarrow \Omega$ such that $\lim_{t\to 0} \gamma(t) = a$. Note that only polynomially bounded o-minimal structures may admit polynomial curve selection. Clearly, if UPC condition holds in $\mathcal{S}$, then it admits polynomial curve selection. We do not know whether the reverse implication is true. The related question is the following: suppose that a bounded and definable set E possesses the property that for each $a \in \overline{E}$ there is a polynomial arc $\gamma : (0,\ 1) \longrightarrow \mathrm{Int}\, E$ such that $\lim_{t\to 0} \gamma(t) = a$—is then E a UPC set?. The example below shows that this is not the case even if we restrict ourselves to polynomially bounded o-minimal structures, but first we state the following lemma:

Lemma 3.1. *Suppose that a bounded set $E \subset \mathbb{R}^n$ is UPC and let $h_x(t) = \sum_{i=0}^{d} a_i(x) t^i$ for $(x,t) \in \overline{E} \times [0,\ 1]$ be any polynomial map satisfying the definition of a UPC set. Then each function $a_i^k : \overline{E} \longrightarrow \mathbb{R}$ is bounded, where $a_i(x) = (a_i^1(x), \ldots, a_i^n(x))$.*

Proof. For each $k \in \{1, \ldots, n\}$ consider the system of linear equations

$$\sum_{i=0}^{d} (j^{-1})^i z_i = h_x^k(j^{-1}), \quad j = 1, \ldots, d+1,$$

where z_i are the unknowns. By Cramer's rule, we get the only solution $z_i = a_i^k(x),\ i = 0, \ldots, d$, bounded, as required. □

Example 3. Let $A = A_1 \cup A_2$, where $A_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^{\sqrt{2}} \leqslant x_2 \leqslant x_1^{\sqrt{2}} + x_1^2,\ 0 \leqslant x_1 \leqslant 1\}$ and $A_2 = [-1, 0] \times [-1, 0]$. Suppose that A is UPC and that $h_x(t)$ is a polynomial map from the definition of UPC. For $n \in \mathbb{N} \setminus \{0\}$, let $h_n(t) = h_{x(n)}(t) = \sum_{i=0}^{d} a_i(n) t^i$, where $x(n) = (n^{-1}, n^{-\sqrt{2}})$. By Lemma 3.1, for each $i = 0, 1, \ldots, d$, the sequence $a_i(n)$ is bounded. We can assume that it is convergent and that $a_i(n) \to a_i$ as $n \to \infty$. Since $\operatorname{dist}(h_n(t), \mathbb{R}^2 \setminus A_1) \geqslant M t^m$, hence $\operatorname{dist}\left(\sum_{i=0}^{d} a_i t^i, \mathbb{R}^2 \setminus A_1\right) \geqslant M t^m$. Clearly, $h(t) := \sum_{i=0}^{d} a_i t^i \in \operatorname{Int} A_1$ for $t \in (0, 1]$, and $h(0) = (0, 0)$. This is a contradiction (cf. Example 2).

We will say that a polynomially bounded o-minimal structure satisfies the property (P) if:

For each definable $f : (0, \varepsilon) \longrightarrow \mathbb{R}$ such that $\lim_{t \to 0} f(t) = 0$ and for each $r > 0$ there exist $c_1, \ldots, c_k \in \mathbb{R}$ and rational $r_1, \ldots, r_k \in (0, +\infty)$ such that $f(t) = c_1 t^{r_1} + \cdots + c_k t^{r_k} + o(t^r)$ as $t \to 0^+$.

Recall now a result due to C. Miller:

Theorem 3.2 (*cf. Miller [8]*). *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be definable in some polynomially bounded o-minimal structure. Then there is $r \in \mathbb{R}$ such that either $f(t) = 0$ for all sufficiently small positive t, or $f(t) = ct^r + o(t^r)$ as $t \to 0^+$ for some $c \in \mathbb{R} \setminus \{0\}$ and the function $(0, +\infty) \ni t \mapsto t^r \in \mathbb{R}$ is definable.*

The following theorem gives a sufficient and necessary condition for a polynomially bounded o-minimal structure to admit polynomial curve selection.

Theorem 3.3. *Let $\mathcal{S}$ be a polynomially bounded o-minimal structure. Then $\mathcal{S}$ admits polynomial curve selection if and only if it satisfies the property (P).*

Proof. Assume then that $\mathcal{S}$ satisfies the property (P). Let $\Omega \subset \mathbb{R}^n$ be open, definable in $\mathcal{S}$ and take $a \in \overline{\Omega}$. We can assume that $a = 0$. By curve selection and the Łojasiewicz inequality there exists a definable map $g = (g^1, \ldots, g^n) : (0, 1) \longrightarrow \Omega$ such that $\lim_{t \to 0} g(t) = 0$ and $\operatorname{dist}(g(t), \mathbb{R}^n \setminus \Omega) \geqslant M t^m$ with some positive constants M, m. If we apply the property (P) for each g^j and $r = m$, then we obtain $c_1, \ldots, c_k \in \mathbb{R}^n$ and rational $r_1, \ldots, r_k \in (0, +\infty)$ such that $\operatorname{dist}\left(\sum_{j=1}^{k} c_j t^{r_j}, \mathbb{R}^n \setminus \Omega\right) > 0$ for t small enough. The existence of the required polynomial curve is now obvious.

Suppose now that $\mathcal{S}$ admits polynomial curve selection. Note first that the map $(0, +\infty) \ni t \mapsto t^s \in \mathbb{R}$ is definable if and only if $s \in \mathbb{Q}$. If we assume that this is not the case, then the set $B = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^s < x_2 < 2x_1^s,\ x_1 \in (0, 1)\}$, for some $s \in (0, +\infty) \setminus \mathbb{Q}$, is definable. This is, however, impossible since there is no polynomial arc $\gamma : (0, 1) \longrightarrow B$ such that $\lim_{t\to 0} \gamma(t) = (0, 0)$. Let $f : (0, \varepsilon) \longrightarrow \mathbb{R}$ be definable and $\lim_{t\to 0} f(t) = 0$. Take any $r > 0$. If $f(t) = 0$ for all sufficiently small positive t, then clearly $f(t) = o(t^r)$ as $t \to 0^+$. If $f(t) \neq 0$ for all sufficiently small positive t, then applying Theorem 3.2 we get $a_1 \in \mathbb{R} \setminus \{0\}$ and $r_1 \in \mathbb{R}$ such that $f(t) = a_1 t^{r_1} + o(t^{r_1})$ as $t \to 0^+$. Obviously, $r_1 > 0$. We do the same thing with $f(t) - a_1 t^{r_1}$. Again either $f(t) - a_1 t^{r_1} = 0$ for all t small enough (and then we stop), or we use Theorem 3.2 getting $f(t) - a_1 t^{r_1} = a_2 t^{r_2} + o(t^{r_2})$ as $t \to 0^+$, where $a_2 \in \mathbb{R} \setminus \{0\}$ and $r_2 \in \mathbb{R}$. Note that $r_2 > r_1$. We continue this process. If it stops at some point, then $f(t) = a_1 t^{r_1} + \cdots + a_k t^{r_k}$ for t small enough and in this case clearly $f(t) = a_1 t^{r_1} + \cdots + a_k t^{r_k} + o(t^r)$ as $t \to 0^+$. If the process does not stop, then we obtain a sequence $\{a_j\}$ of nonzero real numbers and an increasing sequence of positive rationals $\{r_j\}$. Let $\kappa := \lim_{j\to+\infty} r_j \in (0, +\infty) \cup \{+\infty\}$.

Case 1: $\kappa = +\infty$. Then put $l := \min\{j : r_j \geqslant r\}$. Obviously, $f(t) = \sum_{j=1}^{l} a_j t^{r_j} + o(t^r)$ as $t \to 0^+$.

Case 2: $\kappa < +\infty$. Let $\mu \geqslant \kappa$ be rational. Then the set $K = \{(x_1, x_2) \in \mathbb{R}^2 \mid f(x_1) < x_2 < f(x_1) + x_1^{\mu},\ x_1 \in (0, \varepsilon)\}$ is definable, but this contradicts the assumption that $\mathcal{S}$ admits polynomial curve selection, since there is no polynomial arc $\gamma : (0, 1) \longrightarrow K$ such that $\lim_{t\to 0} \gamma(t) = (0, 0)$. □

The above theorem and its proof allow to better understand the meaning of the two examples given in the Introduction and their connection with Theorem 3.2 and the property (P).

4. UPC condition on the plane

In this section, we give a proof of Theorem B which can be regarded as a characterization of bounded and plane UPC sets definable in polynomially o-minimal structures.

Suppose that $A \subset B \subset \mathbb{R}^n$. We say that A is UPC *with respect to* B if there exist positive constants M, m and a positive integer d such that for each point $x \in \overline{A}$ we can choose a polynomial map $h_x : \mathbb{R} \longrightarrow \mathbb{R}^n$ of degree at most d satisfying the following conditions:

(1) $h_x(0) = x$,
(2) $\operatorname{dist}(h_x(t), \mathbb{R}^n \setminus B) \geqslant M t^m$ for all $t \in [0, 1]$.

Clearly, A is UPC if A is UPC with respect to itself.

Lemma 4.1. *Let* $g, h : [0, b] \longrightarrow \mathbb{R}$ *be continuous and definable in some polynomially bounded o-minimal structure,* $g(0) = h(0) = 0$ *and* $h > g$ *on* $(0, b]$. *Suppose that* $\phi_1, \phi_2 : [0, 1] \longrightarrow \mathbb{R}$ *are polynomial functions,* $\phi_1(0) = \phi_2(0) = 0$, $\phi_1([0, 1]) = [0, a]$, $0 < a < b$ *and* $h(\phi_1(s)) > \phi_2(s) > g(\phi_1(s))$ *for* $s \in (0, 1]$. *Let*

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in [0, b],\ g(x_1) \leqslant x_2 \leqslant h(x_1)\}.$$

Then there is a neighbourhood U *of* $(0, 0)$ *in* A *such that* U *is UPC with respect to* A.

Proof. Without loss of generality we may assume that ϕ_1, $h \circ \phi_1 - \phi_2$ are strictly increasing and $g \circ \phi_1 - \phi_2$ is strictly decreasing on $[0, 1]$ (by the monotonicity theorem). Take $x_1 \in \left[0, \frac{a}{2}\right]$ and $x_2 \in [g(x_1), h(x_1)]$. Put

$$W_{(x_1,x_2)}(u) = (\phi_1(u),\ \phi_2(u) + x_2 - \phi_2(\phi_1^{-1}(x_1))), \quad u \in [\phi_1^{-1}(x_1), 1].$$

Note that $g(\phi_1(u)) < \phi_2(u) + x_2 - \phi_2(\phi_1^{-1}(x_1)) < h(\phi_1(u))$ for $u \in (\phi_1^{-1}(x_1), 1]$. Moreover, $W_{(x_1,x_2)}(\phi_1^{-1}(x_1)) = (x_1, x_2)$. Let

$$P_{(x_1,x_2)}(t) = W_{(x_1,x_2)}((1-t)\phi_1^{-1}(x_1) + t), \quad t \in [0, 1].$$

The map $(x_1, x_2, t) \mapsto P_{(x_1,x_2)}(t)$ is continuous and definable, $P_{(x_1,x_2)}(0) = (x_1, x_2)$. Note that $\mathrm{dist}(P_{(x_1,x_2)}(t), \mathbb{R}^2 \setminus A) = 0$ implies $t = 0$. Now it is enough to use the Łojasiewicz inequality. □

Proof of Theorem B. Suppose that A satisfying the assumptions of the theorem is UPC and take $a = (a_1, a_2) \in \overline{A}$, $r > 0$. Let S be any connected component of the set $\mathrm{Int}\, A \cap B(a, r)$ such that $a \in \overline{S}$. Take a cell decomposition of $\mathbb{R}^2$ that partitions $S, A, B(a, r), \{a\}$ and that is minimal with respect to S. This means that if we have two open cells (f, g) and (g, h) contained in S such that the graph $\Gamma(g)$ of g is also contained in S, then replace them by (f, h).

Let $C \subset S$ be an open cell such that $a \in \overline{C}$. Suppose that there is no polynomial arc $\gamma : (0, 1) \longrightarrow C$ such that $\lim_{t\to 0} \gamma(t) = a$. Let $\varphi, \psi : (\alpha, \beta) \longrightarrow \mathbb{R}$ be such that $C = (\varphi, \psi)$. Clearly, $a_1 = \alpha$ or $a_1 = \beta$. We may assume that $a_1 = \alpha$. It is easy to see that $\lim_{t\to\alpha} \varphi(t) = \lim_{t\to\alpha} \psi(t) = a_2$ (since the o-minimal structure is polynomially bounded). Put $K := \{\beta\} \times [s_1, s_2]$, where $s_1 = \lim_{t\to\beta} \varphi(t)$ and $s_2 = \lim_{t\to\beta} \psi(t)$ (if $s_1 = s_2$, then $[s_1, s_2] := \{s_1\}$). Take any $\eta \in (\alpha, \beta)$ and put $L := \{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_2 = \varphi(x_1) + \psi(x_1),\ x_1 \in (\alpha, \eta)\}$. For the set A we choose h, m, M from the definition of a UPC set. Note first that there is some constant $\theta \in (0, 1]$ such that $\|h_x(t) - x\| \leqslant \frac{c}{2}$ for all $x \in \overline{A}$ and $t \in [0, \theta]$, where $c := \mathrm{dist}(L, K)$ (it follows from Lemma 3.1). Thus $\mathrm{dist}(h_x(t), K) \geqslant \frac{c}{2}$ for $x \in L$ and $t \in [0, \theta]$. Let C_1, C_2 denote the two open cells that lie, respectively, just below and just above the cell C. Note that $\Gamma(\varphi)$ is disjoint from A or C_1 is disjoint from A. Similarly $\Gamma(\psi)$ is disjoint from A or C_2 is disjoint from A. Assume, for example, that $\Gamma(\varphi) \subset A$ and $C_1 \subset A$. One can easily see that the open and connected set $C_1 \cup \Gamma(\varphi) \cup C$ is contained in S. This is, however impossible, since our cell decomposition is minimal with respect to S.

All this easily implies that $\operatorname{dist}(h_x(t),\ \mathbb{R}^2 \setminus C) \geqslant \min\left\{Mt^m,\ \frac{c}{2}\right\} \geqslant M't^m$ for $x \in L$ and $t \in [0,\ \theta]$, where M' is some positive constant. Now the situation is essentially the same as in Example 3 and it is enough to use the same reasoning to get a contradiction.

The reverse implication in Theorem B follows from Lemma 4.1. □

Remark. It follows from Theorems B and 3.3 that if a polynomially bounded o-minimal structure satisfies the condition (P), then any bounded, fat and definable subset $A \subset \mathbb{R}^2$ is UPC.

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